The hidden symmetry-breaking picture of symmetry-protected topological order

Dominic V. Else,[1,2] Stephen D. Bartlett,[1] and Andrew C. Doherty[1]

[1]*Centre for Engineered Quantum Systems, School of Physics,*
The University of Sydney, Sydney, NSW 2006, Australia
[2]*Department of Physics, University of California, Santa Barbara, CA 93106, USA*

We generalize the hidden symmetry-breaking picture of symmetry-protected topological (SPT) order developed by Kennedy and Tasaki in the context of the Haldane phase. Our generalization applies to a wide class of SPT phases in one-dimensional spin chains, protected by an on-site representation of a finite abelian group. This generalization takes the form of a non-local unitary map that relates local symmetry-respecting Hamiltonians in an SPT phase to local Hamiltonians in a symmetry-broken phase. Using this unitary, we establish a relation between the two-point correlation functions that characterize fully symmetry-broken phases with the string-order correlation functions that characterise the SPT phases, therefore establishing the perspective in these systems that SPT phases are characterised by *hidden* symmetry-breaking. Our generalization is also applied to systems with continuous symmetries, including $SO(2k+1)$ and $SU(k)$.

I. INTRODUCTION

In the traditional Landau paradigm, order in condensed matter systems is viewed as synonymous with the spontaneous breaking of a symmetry. However, it is now well-established that at zero temperature there exist topologically ordered phases, such as those of the fractional quantum Hall effect, which cannot be understood through the Landau paradigm. Additionally, one can also consider *symmetry-protected topological (SPT)* phases[1], which are not topologically ordered in the conventional sense, yet remain distinct from the trivial phase in the presence of an appropriate symmetry. A well-known example of an SPT phase is the Haldane phase of antiferromagnetic spin-1 chains, which is protected by the $D_2 \cong Z_2 \times Z_2$ symmetry comprising π rotations about a set of orthogonal axes. The "topological" nature of the Haldane phase is manifested in a number of ways, such as the long-range string order[2], fractionalized edge modes[3], degenerate entanglement spectrum[4], and long-range localizable entanglement[5,6]. It is now known that many of the interesting properties of the Haldane phase extend in general to SPT phases of one-dimensional spin chains protected by a unitary "on-site" representation of an arbitrary symmetry group G (for which the Haldane phase, with $G = Z_2 \times Z_2$, is the simplest non-trivial example), which have been fully classified[7,8].

An early and influential characterization of the Haldane phase was provided by Kennedy and Tasaki[9] (see also Ref. 10). They constructed a non-local unitary [which we refer to as the *Kennedy-Tasaki (KT) transformation*] to transform the Haldane phase to a conventional symmetry-breaking phase. Although the transformation is non-local, for any local Hamiltonian H which respects the D_2 rotation symmetry, the KT transformation yields another D_2-symmetric local Hamiltonian $\widetilde{H}$. It turns out that if H is in the SPT Haldane phase, then $\widetilde{H}$ will have a space of four degenerate locally-distinguishable

ground states corresponding to the spontaneous breaking of the D_2 rotation symmetry, i.e. $\widetilde{H}$ is in the maximal symmetry-breaking phase for the D_2 symmetry. Thus, the ordering in the SPT phase is interpreted as "hidden symmetry-breaking". Furthermore, the long-ranged string order in the Haldane phase is related by the KT transformation to conventional long-ranged order in the symmetry breaking phase.

The Haldane phase is also a special case of the SPT phases that were classified in Refs. 7 and 8 through the symmetry properties of a matrix-product state ansatz for the ground state. The general result is that the distinct SPT phases with respect to an on-site representation of a symmetry group G are classified by the second cohomology classes of the projective representations of the symmetry group. This classification appears to be the most fundamental and general approach to SPT phases for one-dimensional chains with an on-site symmetry. However, the connection with the original hidden symmetry-breaking picture of Kennedy and Tasaki has not been explored. As a result, it remains unclear to what extent other SPT phases can also be understood to arise from a similar hidden symmetry-breaking mechanism (although see Ref. 11 for one example).

In this paper, we extend the hidden-symmetry breaking picture to any SPT phase protected in a one-dimensional spin chain by an on-site representation of a finite abelian group, provided that the cohomology class describing the phase satisfies a condition called *maximal non-commutativity*. (We say that a cohomology class is maximally non-commutative if, in the corresponding projective representations, for the matrix representation of any non-trivial group element there exists at least one other matrix in the representation with which it does not commute[12].) We achieve this by constructing a suitable generalization of the KT transformation (presented in Sec. IV), expressed explicitly in terms of the appropriate cohomology class of the symmetry group, to transform the SPT phase into a conventional symmetry-breaking

phase. The generalized KT transformation is essentially equivalent to the duality transformation introduced by us in the context of quantum computation[13], and some of its properties were already discussed in the appendices of that paper; however, our treatment here will be self-contained.

Where it can be applied, our generalized KT transformation affords a different perspective on properties of the SPT phase. For example, for abelian symmetry groups, SPT phases (and indeed, all symmetric phases) can be identified from a pattern of string order (as we will show, based in part on the results of Ref. 14). For an SPT phase corresponding to a maximally non-commutative cohomology class, this pattern of string order can be understood in a natural way through the generalized KT transformation, which relates it to the long-range order characterizing the symmetry-breaking phase (just as the original KT transformation does in the case of the Haldane phase). We explore this perspective in Sec. VI. We remark that, although we are only able to consider finite abelian symmetry groups, these groups can arise as subgroups for systems with a larger symmetry. For SPT phases in systems that have a $SO(2k+1)$ or $SU(k)$ symmetry, in Sec. VII we will exhibit an appropriate finite abelian subgroup which allows the hidden symmetry breaking to be identified.

II. THE KENNEDY-TASAKI TRANSFORMATION

Let us recall the definition of the unitary $\mathcal{D}_{KT}$ which effects the Kennedy-Tasaki transformation for a chain of N spin-1's with open boundary conditions. It can be written as[15]

$$\mathcal{D}_{KT} = \prod_{j<k} \exp(i\pi S_j^z S_k^x), \qquad (1)$$

where S_j^a ($a = x, y, z$) denotes the appropriate spin component operator for the j-th spin. This unitary is non-local, but, for any local observable A which respects the D_2 symmetry operations $\prod_j \exp(i\pi S_j^a)$ ($a = x, y, z$), the transformed observable $\mathcal{D}_{KT} A \mathcal{D}_{KT}^\dagger$ remains local and symmetry-respecting. Therefore, for any Hamiltonian H that is the sum of local symmetry-respecting interactions, one can generate the dual Hamiltonian $\widetilde{H} = \mathcal{D}_{KT} H \mathcal{D}_{KT}^\dagger$. If H is in the SPT phase with respect to the D_2 symmetry, then $\widetilde{H}$ is expected to be in a conventional symmetry-breaking phase with respect to the symmetry, with the four-fold degenerate edge states mapping under $\mathcal{D}_{KT}$ to the four locally distinguishable symmetry-breaking ground states.

In order to see that $\mathcal{D}_{KT}$ is a special case of the generalized KT transformation to be defined later, we will want to express $\mathcal{D}_{KT}$ in terms of the single-site basis $\{|x\rangle, |y\rangle, |z\rangle\}$ (where $|a\rangle$ is the zero eigenstate of S^a for $a = x, y, z$), which is the simultaneous eigenbasis of the

on-site representation of the symmetry. Observe that

$$\exp(i\pi S_z \otimes S_x)(|a_1\rangle \otimes |a_2\rangle) = (-1)^{\mu(a_1)\nu(a_2)}|a_1\rangle \otimes |a_2\rangle, \qquad (2)$$

where $\mu(a) = 1 - \delta_{a,z}$ and $\nu(a) = 1 - \delta_{a,x}$. Hence we can write

$$\mathcal{D}_{KT} = \sum_{a_1,\ldots,a_N} (-1)^{\sum_{j<k} \mu(a_j)\nu(a_k)}|a_1,\ldots,a_N\rangle\langle a_1,\ldots,a_N|. \qquad (3)$$

III. CLASSIFICATION OF SPT PHASES BY COHOMOLOGY CLASSES

Throughout this paper, we will assume a chain of N spins, such that the Hamiltonian commutes with the on-site representation $[u(g)]^{\otimes N}$ of a symmetry group G. We will assume open boundary conditions unless otherwise stated. According to the general classification of SPT phases for on-site symmetries in one-dimensional systems[7,8], the different SPT phases for this symmetry can be classified by the second cohomology group $H^2(G, U(1))$, which is related to the projective representations $V(g)$ for the group G, as we now describe. (One interpretation of these projective representations is that they describe the action of the symmetry on the fractionalized edge mode associated with a physical edge.) By definition, a projective representation $V(g)$ must satisfy

$$V(g_1)V(g_2) = \omega(g_1, g_2)V(g_1 g_2), \quad \forall g_1, g_2 \in G, \qquad (4)$$

where ω is a function mapping pairs of group elements to complex phase factors, known as the *factor system* of the projective representation. The associativity of matrix multiplication implies that the factor system must satisfy the 2-cocycle condition

$$\omega(g_1, g_2)\omega(g_1 g_2, g_3) = \omega(g_2, g_3)\omega(g_1, g_2 g_3),$$
$$\forall g_1, g_2, g_3 \in G. \quad (5)$$

Conversely, any ω satisfying Eq. (5) is the factor system for some projective representation[16]. Furthermore, given any projective representation, it is trivial to generate another one by rephasing of the operators $V(g)$, i.e. $V(g) \to \beta(g)V(g)$, where β is a function which sends group elements to phase factors. The effect on the factor system is

$$\omega(g_1, g_2) \to \beta(g_1 g_2)^{-1}\beta(g_1)\beta(g_2)\omega(g_1, g_2). \qquad (6)$$

Two factor systems related by a transformation of the form Eq. (6) are said to be in the same *cohomology class*, and the second cohomology group $H^2(G, U(1))$ comprises all the distinct cohomology classes for the group G. We will denote by $[\omega]$ the cohomology class containing a given factor system ω.

In the case of the Haldane phase for spin-1 chains, the relevant symmetry group is $D_2 = \{1, x, y, z\}$ (where $y =$

xz), with the on-site representation $u(a) = \exp(i\pi S_a)$ (for $a = x, y, z$). The Haldane phase corresponds to the unique non-trivial cohomology class for the symmetry group D_2. We can specify a representative factor system ω for this cohomology class by giving an example of a projective representation for which ω is the factor system, namely

$$V(1) = \mathbb{I}, \quad V(x) = \sigma_x, \quad V(z) = \sigma_z, \quad V(y) = \sigma_x\sigma_z, \tag{7}$$

where σ_x and σ_z are the respective Pauli spin matrices. We define $V(y)$ as above, rather than the more symmetrical $V(y) = \sigma_y$ (which would correspond to a different factor system within the same cohomology class), because the factor system of Eq. (7) will turn out to be closely connected to the conventional formulation of the KT transformation.

IV. THE GENERALIZED KENNEDY-TASAKI TRANSFORMATION

In this section, we will define our generalized KT transformation, for an SPT phase characterized by a cohomology class $[\omega]$ and an on-site symmetry representation of a group G. We will require that G be finite and abelian, and that the cohomology class $[\omega]$ be maximally non-commutative (to be defined below). A special property of an abelian symmetry is that the irreps are one-dimensional; therefore, the on-site representation $u(g)$ must decompose as

$$u(g) = \bigoplus_\chi \chi(g)\mathbb{I}_{m_\chi} \tag{8}$$

where the sum is over the one-dimensional representations (characters) χ of G. For simplicity of presentation, we assume that none of the multiplicities m_χ are greater than 1; thus, we can write

$$u(g) = \sum_\chi \chi(g)|\chi\rangle\langle\chi| \tag{9}$$

where the $\{|\chi\rangle\}$ form an orthonormal basis, and the sum is over those χ such that $m_\chi > 0$. However, all the results of this paper can easily be generalized to the case of multiplicities greater than 1. For the Haldane phase, this basis $\{|\chi\rangle\}$ is the basis $\{|x\rangle, |y\rangle, |z\rangle\}$ discussed in Sec. II.

Our generalized construction applies for any SPT phase with respect to the aforementioned symmetry, so long as the corresponding cohomology class $[\omega]$ is *maximally-noncommutative*, which is to say that the subgroup $G(\omega) = \{g \in G : V_\omega(g)V_\omega(h) = V_\omega(h)V_\omega(g) \; \forall \; h \in G\}$ is trivial. This property does not depend on the choice of representative factor system for the cohomology class. (Throughout this section, we will use V_ω to denote some projective representation of G with factor system ω; it does not matter how the projective representation is chosen because we only use the multiplicative relations between the matrices $V_\omega(g)$, and these are

determined by ω.) An important property of a maximally non-commutative factor system is the following. Any cohomology class for an abelian group can be considered to induce a homomorphism φ_ω from G to G^* (where G^* is the *character group* of G, i.e. the group of one-dimensional projective representations of G under multiplication), according to

$$\varphi_\omega(g) = \chi_g^\omega, \tag{10}$$

where χ_g^ω is the one-dimensional representation of G such that

$$V_\omega(g')V_\omega(g)V_\omega(g')^\dagger = \chi_g^\omega(g')V_\omega(g). \tag{11}$$

Observe that from Eq. (11) one can prove both that $\chi_g^\omega(g_1')\chi_g^\omega(g_2') = \chi_g^\omega(g_1'g_2')$ (i.e. $\chi_g^\omega = \varphi_\omega(g)$ is in G^*) and that $\chi_{g_1}^\omega\chi_{g_2}^\omega = \chi_{g_1g_2}^\omega$ (i.e. φ_ω is a homomorphism). For the particular case of a maximally non-commutative projective representation of a finite abelian group, the kernel of φ_ω [which is equal to $G(\omega)$ in general] is trivial, and therefore φ_ω is an *iso*morphism; that is, for any $\chi \in G^*$ there is a unique $g \in G$ such that $\chi_g^\omega = \chi$.

We construct the unitary $\mathcal{D}_\omega$ corresponding to the generalized Kennedy-Tasaki transformation (as we will see later, it maps *from* the maximal symmetry-breaking phase into the SPT phase), acting on a chain of N sites with open boundary conditions, according to

$$\mathcal{D}_\omega = \sum_{\boldsymbol{\chi}} \Omega_\omega\big(\varphi_\omega^{-1}(\boldsymbol{\chi})\big)|\boldsymbol{\chi}\rangle\langle\boldsymbol{\chi}|, \tag{12}$$

where we use the abbreviations $\boldsymbol{\chi} = (\chi_1, \ldots, \chi_N)$, $\varphi_\omega^{-1}(\boldsymbol{\chi}) = \big(\varphi_\omega^{-1}(\chi_1), \ldots, \varphi_\omega^{-1}(\chi_N)\big)$, and $\Omega_\omega(\mathbf{g})$ is the phase factor defined such that

$$V_\omega(g_N)\cdots V_\omega(g_1) = \Omega_\omega\big(\mathbf{g}\big)V_\omega(g_N\cdots g_1). \tag{13}$$

For the case of a spin-1 chain with D_2 symmetry, one can check directly that the choice of factor system ω defined by the projective representation Eq. (7) gives

$$V(a_N)\cdots V(a_1) = (-1)^{\sum_{j<k} \mu(a_j)\nu(a_k)}V(a_N\cdots a_1), \tag{14}$$

(where, loosely, one obtains a phase factor of -1 for every $V(z) = \sigma_z$ operator to the left of a $V(x) = \sigma_x$ operator). Therefore, Eq. (12) reduces to the standard Kennedy-Tasaki transformation Eq. (3) if we choose this factor system. Note that the definition of $\mathcal{D}_\omega$ is not the same for different factor systems ω within the same cohomology class. However, the difference is not very significant; see Appendix A.

Due to the way the unitary $\mathcal{D}_\omega$ is defined, we can immediately derive the basic property that, although it is a non-local transformation, for any *symmetry-respecting* observable A supported on a block of n sites, the transformed observable $\mathcal{D}A\mathcal{D}^\dagger$ is still supported on the same block. We will use the notation $\boldsymbol{\chi} = (\boldsymbol{\chi}_l, \boldsymbol{\chi}_b, \boldsymbol{\chi}_r)$, corresponding to grouping the sites in the chain according to whether they are, respectively, to the left of, within,

or to the right of the block containing the support of A. Thus the matrix element $\langle \boldsymbol{\chi}_l, \boldsymbol{\chi}_b, \boldsymbol{\chi}_r | A | \boldsymbol{\xi}_l, \boldsymbol{\xi}_b, \boldsymbol{\xi}_r \rangle$ can be nonzero only if $\boldsymbol{\chi}_l = \boldsymbol{\xi}_l$, $\boldsymbol{\chi}_r = \boldsymbol{\xi}_r$, and $\prod_{j=1}^n \chi_{b,j} = \prod_{j=1}^n \xi_{b,j}$ (the last condition comes from the assumption that A commutes with the symmetry). As a result, it is easy to show from the definition of Ω [Eq. (13)] that $\Omega_\omega\big(\varphi_\omega^{-1}(\boldsymbol{\chi})\big) \ \Omega_\omega\big(\varphi_\omega^{-1}(\boldsymbol{\xi})\big)^{-1} = \Omega_\omega^{(n)}\big(\varphi_\omega^{-1}(\boldsymbol{\chi}_b)\big)\Omega_\omega^{(n)}\big(\varphi_\omega^{-1}(\boldsymbol{\xi}_b)\big)^{-1}$, and hence that $\mathcal{D}_\omega A \mathcal{D}_\omega^\dagger = \mathcal{D}_\omega^{(n)} A \mathcal{D}_\omega^{(n)\dagger}$, where $\Omega_\omega^{(n)}$ and $\mathcal{D}_\omega^{(n)}$ are defined as Ω_ω and $\mathcal{D}_\omega$ would be if the n sites in the block constituted the entire chain.

V. ACTION OF THE GENERALIZED KT TRANSFORMATION ON A GENERALIZED AKLT STATE

Although we have focussed on the transformation of the Hamiltonian under the generalized KT transformation $\mathcal{D}_\omega$, for illustrative purposes we will consider in this section a particular Hamiltonian within the SPT phase described by cohomology class $[\omega]$, for which the ground state subspace can be found analytically. We calculate explicitly how this subspace transforms under $\mathcal{D}_\omega^\dagger$, and show that the transformed ground state subspace reflects the spontaneous breaking of the symmetry in the bulk. The definition of $\mathcal{D}_\omega$ [Eqs. (12) and (13)] arises naturally out of this discussion.

Affleck, Kennedy, Lieb and Tasaki (AKLT)[17] constructed a system in the Haldane phase for which the ground state can be represented exactly as a "valence-bond solid", or (in more modern language) a *matrix-product state* (MPS)[18]. We will now define a generalization of the AKLT ground state for the SPT phase with cohomology class $[\omega]$. We will write it for open boundary conditions, which means we have to define a subspace $\mathcal{P}$ of states corresponding to the degenerate ground-state subspace. The states in this subspace are of the MPS form

$$\sum_{\boldsymbol{\chi}} \mathrm{Tr}(A_{\chi_N} \cdots A_{\chi_1} B) | \boldsymbol{\chi} \rangle, \tag{15}$$

where we set $A_\chi = V_\omega\big(\varphi_\omega^{-1}(\chi)\big)$ (where V_ω is an irreducible projective representation with factor system ω), and the subspace comprises the states obtained from all possible $D \times D$ matrices B (with D the dimension of V_ω). The theory of MPS parent Hamiltonians[18] allows one to construct a local frustration-free Hamiltonian for which $\mathcal{P}$ is the D^2-fold degenerate ground state subspace. From the classification of SPT order in matrix-product states[7,8], one can show that H indeed lies in the SPT phase described by cohomology class $[\omega]$.

We will only consider the state $|\Psi\rangle$ resulting from setting $B = \frac{1}{D}\sum_g V(g)^\dagger$, as it turns out that applying $[u(g)]^{\otimes N}$ to $|\Psi\rangle$ for group elements g generates a basis

for $\mathcal{P}$. This gives

$$|\Psi\rangle = \sum_{\boldsymbol{\chi}} \Omega_\omega\big(\varphi_\omega^{-1}(\boldsymbol{\chi})\big)|\boldsymbol{\chi}\rangle, \tag{16}$$

where we have used the fact that $\mathrm{Tr}[V(h)V(g)^\dagger] = D\delta_{g,h}$ (which follows from the fact that $\mathrm{Tr}\, V(g) = 0$ for $g \neq 1$, a consequence of maximal non-commutativity). This means that

$$\mathcal{D}_\omega^\dagger |\Psi\rangle = \sum_{\boldsymbol{\chi}} |\boldsymbol{\chi}\rangle = |\phi\rangle^{\otimes N}, \tag{17}$$

where $|\phi\rangle = \sum_\chi |\chi\rangle$. Since $[u(g)]^{\otimes N}$ commutes with $\mathcal{D}_\omega^\dagger$, a basis for the the transformed subspace $\mathcal{D}_\omega^\dagger\mathcal{P}$ comprises the states $\{[u(g)]^{\otimes N}|\phi\rangle^{\otimes N}, g \in G\}$. Thus the transformed Hamiltonian under $\mathcal{D}_\omega^\dagger$ indeed has a set of locally distinguishable symmetry-breaking ground states, as we expect.

VI. STRING ORDER

A key property of the Kennedy-Tasaki transformation is that it relates two-particle correlations (which are expected to be long-ranged in the maximal symmetry-breaking phase for the $Z_2 \times Z_2$ symmetry) to the string correlation functions that characterize the Haldane phase. Here, we will establish a similar correspondence for our general construction. This property will also allow us to determine how the generalized Kennedy-Tasaki transformation maps between different quantum phases.

A. Symmetry-breaking phases and two-particle correlations

Let us first give a general discussion of the two-particle correlations which we expect to see in the maximal symmetry-breaking phase (i.e. where the subgroup of symmetry operations which are *not* spontaneously broken in the bulk is trivial) for an on-site abelian symmetry. A system in this phase will have a collection of degenerate symmetry-breaking ground states. Traditionally, the symmetry-breaking is detected through the nonzero value of an *order parameter*, which is the expectation value of a single-site observable A such that $\langle A \rangle = 0$ for any symmetry-*respecting* state. For example, in the case of the quantum transverse-field Ising model on spin-1/2's, with Hamiltonian

$$H = -\sum_i \sigma_i^z \sigma_{i+1}^z + \lambda \sum_i \sigma_i^x, \tag{18}$$

the appropriate order parameter is $\langle \sigma_z \rangle$. The fact that $\sigma_x \sigma_z \sigma_x = -\sigma_z$ ensures that $\langle \sigma_z \rangle$ must be zero for any state respecting the spin-flip symmetry $\prod_i \sigma_i^x$. In the general case, we can consider an observable A such that

$$u(g')Au(g')^\dagger = \chi(g')A \tag{19}$$

for some $\chi \in G^*$. By a similar argument as before, for $\chi \neq 1$ we find that $\langle A \rangle = 0$ for any symmetry-respecting state. As another example, in the case of a spin-1 chain with the D_2 rotation symmetry, the spin-component operators S_a ($a = x, y, z$) satisfy Eq. (19) for appropriate choices of χ.

Denote the space of operators A satisfying Eq. (19) by $\mathcal{A}_\chi$. For a given symmetry-breaking state, a given operator in $\mathcal{A}_\chi$ could still have zero expectation value by accident. However, we will now argue that, for a given maximal symmetry-breaking state, for *every* nontrivial $\chi \in G^*$ a generic choice of $A \in \mathcal{A}_\chi$ will reveal the symmetry-breaking through its nonzero expectation value. Indeed, let ρ be the reduced state density operator on a single state. It suffices to show that the subspace $\mathcal{B}_\chi = \{ A \in \mathcal{A}_\chi | \mathrm{Tr}(A\rho) = 0 \}$ is a proper subspace (i.e. $\mathcal{B}_\chi \neq \mathcal{A}_\chi$). Suppose by way of contradiction that $\mathcal{B}_\chi = \mathcal{A}_\chi$. Then *every* $A \in \mathcal{A}_\chi$ must satisfy $\mathrm{Tr}(A\rho) = 0$. But since the set $\{ |\chi'\rangle\langle\chi'\chi| : \chi' \in G^* \}$ comprises a basis for $\mathcal{A}_\chi$, this would imply that $\langle\chi'\chi|\rho|\chi'\rangle = 0$ for all χ'. If ρ breaks all the symmetries then there is no reason to expect all these matrix elements to be zero. Certainly, for a *generic* state in the maximal symmetry-breaking phase, this should not be the case.

An alternative measure of the symmetry-breaking is the two-particle correlation function $\langle C_n(A, B) \rangle$ (for $A, B \in \mathcal{A}_\chi$), where

$$C_n(A, B) \equiv A^\dagger \otimes \mathbb{I}^{\otimes(n-2)} \otimes B. \qquad (20)$$

Because each of the symmetry-breaking ground states should be short-range correlated, the correlation function $\langle C_n(A, B) \rangle$ converges to $\langle A \rangle^* \langle B \rangle$ as $n \to \infty$. The expectations here are taken with respect to a particular choice of symmetry-breaking ground state, but notice that $C_n(A, B)$ commutes with the symmetry, and therefore its expectation is independent of this choice.

B. SPT phases and string correlation functions

A key feature of the Haldane phase and its generalizations is that there is no locally-detectable symmetry breaking in the bulk, and consequently all the two-particle correlations $\langle C_n(A, B) \rangle$ decay exponentially as $n \to \infty$. Nevertheless, such SPT phases still have a more subtle form of long-range order detectable through *string* correlations. As a result, we are led to consider the following generalization of Eq. (20) (reducing to it when $g = 1$):

$$C_n(A, B; g) = A^\dagger \otimes [u(g)]^{\otimes(n-2)} \otimes B. \qquad (21)$$

(Recall that $u(g)$ is the unitary on-site action of the symmetry.) In particular, the den Nijs-Rommelse string operators[2] for the Haldane phase,

$$S^\alpha \otimes [e^{i\pi S^\alpha}]^{\otimes(n-2)} \otimes S^\alpha = C_n(S^\alpha, S^\alpha; \alpha), \quad \alpha = x, y, z, \qquad (22)$$

with S^α the spin-component operators, are of this form. In the Haldane phase, the den Nijs-Rommelse string correlations are long-ranged, i.e. $\lim_{n\to\infty} \langle C_n(S^\alpha, S^\alpha; \alpha) \rangle \neq 0$.

Traditionally, long-ranged string correlations have been viewed as evidence of non-trivial order. However, there is a need for caution: the limiting string correlation functions $\lim_{n\to\infty} \langle C_n(A, B; g) \rangle$ are nonzero for generic choices of A and B whenever the symmetry is unbroken in the bulk, and need not reflect any non-trivial SPT order[14,19]. Therefore, in order to obtain useful criteria for identifying SPT phases, we must restrict ourselves to restricted classes of A and B. Indeed, it turns out to be useful to require, as in the symmetry-breaking case, $A, B \in \mathcal{A}_\chi$ for some linear character χ. In that case, the selection rule discussed in Ref. 14 forces $\lim_{n\to\infty} \langle C_n(A, B; g) \rangle = 0$ when $\varphi_\omega(g) \neq \chi$, where φ_ω is the homomorphism induced by the cohomology class [Eq. (10)] (see Appendix C for the proof). In the case $\varphi_\omega(g) = \chi$, there is no such selection rule and so we expect that the corresponding string correlation will generically be long-ranged. (It can be checked that the latter case is the relevant one for the den Nijs-Rommelse string correlations in the Haldane phase.) Thus, the pattern of long-ranged string orders of the form considered is a useful way of identifying phases; we will make this idea more precise in Section VI D.

C. Mapping of correlation functions under the generalized Kennedy-Tasaki transformation

We have established that the string operators $C_n(A, B; g)$ are useful tools for identifying phases. Therefore, it makes sense to calculate how these operators transform under the generalized KT transformation $\mathcal{D}_\omega$, where $[\omega]$ is a maximally non-commutative cohomology class. This calculation is done in Appendix B; the result is (for $A, B \in \mathcal{A}_\chi$)

$$\mathcal{D}_\omega C_n(A, B; g) \mathcal{D}_\omega^\dagger = C_n(\widetilde{A}, \widetilde{B}; \widetilde{g}) \qquad (23)$$

with

$$\widetilde{A} = A W_\omega(\chi)^\dagger, \quad \widetilde{B} = B W_\omega'(\chi)^\dagger, \quad \widetilde{g} = g\varphi_\omega^{-1}(\chi), \quad (24)$$

where

$$W_\omega(\chi) = \sum_{\chi'} \omega\big(\varphi_\omega^{-1}(\chi), \varphi_\omega^{-1}(\chi')\big) |\chi'\rangle\langle\chi'|, \qquad (25)$$

$$W_\omega'(\chi) = \sum_{\chi'} \omega\big(\varphi_\omega^{-1}(\chi'), \varphi_\omega^{-1}(\chi)\big) |\chi'\rangle\langle\chi'|, \qquad (26)$$

(observe that $A, B \in \mathcal{A}_\chi$ implies $\widetilde{A}, \widetilde{B} \in \mathcal{A}_\chi$ as well, since W_ω and W_ω' commute with the symmetry).

To see the significance of this result, suppose that $[\omega]$ is a maximally non-commutative cohomology class. Recall that, in a system with maximal symmetry breaking, for any $\chi \neq 1$ we expect to be able to find

$A, B \in \mathcal{A}_\chi$ such that $\lim_{n\to\infty}\langle C_n(A, B; 1)\rangle \neq 0$. But then, by Eq. (23), this implies that in the transformed system obtained from the original one by $\mathcal{D}_\omega$, we will have $\lim_{n\to\infty}\langle C_n(\widetilde{A}, \widetilde{B}; \varphi_\omega^{-1}(\chi))\rangle \neq 0$, i.e. there are long-ranged string correlations of precisely the form that we expect to get in the SPT phase characterized by cohomology class $[\omega]$. In the next subsection, we will turn this into a proof that the transformed system is indeed in that phase.

D. Patterns of string order as a "signature" for quantum phases

We have already seen that the long-range behavior of string correlations of the form $\langle C(A, B; g)\rangle$ (with $A, B \in \mathcal{A}_\chi$ for some character χ) is a useful probe for identifying different kinds of ordering in systems with a finite abelian on-site symmetry. In Appendix C, we go further, and show that this long-range behavior *uniquely* identifies all possible quantum phases that result from symmetry-respecting Hamiltonians. (The general classification of such phases was given in Ref. 8 and 20; it includes conventional symmetry-breaking phases, SPT phases with no symmetry-breaking in the bulk, as well as other examples in which SPT and symmetry-breaking orders combine.)

The result of Appendix A is expressed in terms of the following "signature" function M acting on $G^* \times G$ to measure which of the string correlations are long-ranged:

$$M(\chi, g) = \begin{cases} 1 & \text{if } \lim_{n\to\infty}\langle C_n(A, B, g)\rangle \neq 0 \text{ } generically \\ & \text{in the phase, when } A, B \in \mathcal{A}_\chi, \\ 0 & \text{otherwise} \end{cases}$$

$$(27)$$

We have included the word "generically", because it is possible that there might be specific points in the phase and/or choices of $A, B \in \mathcal{A}_\chi$ such that the limiting correlation is "accidentally" zero. [For example, in the case $g = 1$, we derived in Sec. VI A the condition for a given state in the maximal symmetry-breaking phase to satisfy $\lim_{n\to\infty} C_n(A, B; 1)\rangle = 0$ for all $A, B \in \mathcal{A}_\chi$, even though generically we expect these two-body correlations to be long-ranged.] The result of Appendix C is then that each possible phase in the general classification has a distinct signature M.

Combining this result with that of the previous subsection VI C allows us to definitively establish in general how different phases are transformed into each other by the generalized KT transformation $\mathcal{D}_\omega$ (with $[\omega]$ a maximally non-commutative cohomology class). Indeed, suppose we start from a phase described by signature M. Then, by Eqs. (23) and (24), the transformed phase resulting from application of $\mathcal{D}_\omega$ has signature $M'(\chi, g) = M(\chi, g[\varphi_\omega^{-1}(\chi)]^{-1})$.

In particular, we can consider the case that the starting phase is the maximal symmetry breaking phase (all the symmetries broken in the bulk). The arguments of

Sec. VI A show that for such a phase, $M(\chi, 1) = 1$ for all χ. Furthermore, it was shown in Ref. 14 that string order of the form $\langle C_n(A, B; g)\rangle$ can be long-ranged only when the symmetry corresponding to g is unbroken in the bulk. Thus, the maximal symmetry-breaking phase has signature $M(\chi, g) = 1 \Leftrightarrow g = 1$. It follows that the transformed phase resulting from applying $\mathcal{D}_\omega$ has signature $M'(\chi, g) = 1 \Leftrightarrow g = \varphi_\omega^{-1}(\chi)$. From the discussion of Appendix C, we see that this is precisely the signature of the SPT phase with cohomology class $[\omega]$, as expected.

VII. THE KENNEDY-TASAKI TRANSFORMATION FOR CONTINUOUS SYMMETRIES

We stress that our assumption of a maximally-noncommutative cohomology class of a finite abelian group might not be as restrictive as it sounds. Indeed, an SPT phase characterized by an arbitrary group could still be identified as part of a maximally non-commutative SPT phase with respect to a finite abelian subgroup. As an example, here we will discuss how our framework allows us to apply the concept of hidden symmetry breaking to some generalizations of the Haldane phase. Just as the Haldane phase is motivated by SO(3)-invariant antiferromagnets, these generalized Haldane phases contain systems which are invariant under an SO($2k+1$) or SU(k) symmetry. However, in each case, we will identify a finite abelian subgroup (analogous to D_2 for the Haldane phase), which will turn out to be the relevant one for identifying the hidden symmetry breaking. In each case, this finite abelian subgroup will turn out to be sufficient to classify the phases, since imposing the full continuous symmetry does not separate any phases which could not already be distinguished through this subgroup. This suggests that, even when the full continuous symmetry is present, we should describe the SPT order in terms of the hidden breaking of the finite abelian subgroup.

A. The SO($2k + 1$) Haldane phase

For systems invariant under an on-site SO($2k+1$) symmetry, there is exactly one non-trivial SPT phase[11,21], which we can think of as a generalization of the Haldane phase (reducing to it in the case $k = 1$). The corresponding cohomology class is that of the spinor representations of SO($2k + 1$) (which are, in fact, *projective* representations).

Identifying SO($2k+1$) with its representation in terms of $(2k + 1) \times (2k + 1)$ orthogonal matrices with unit determinant, we define $G_k \equiv \{A \in \text{SO}(2k + 1) : A \text{ is diagonal in the standard basis}\}$, which constitutes a finite abelian subgroup. We can construct a minimal set of generators $\{u^{(l)}, l = 1, \ldots, 2k\}$ with matrix elements

$$u_{i,j}^{(l)} = (-1)^{1-\delta_{i,l}}\delta_{i,j} \qquad (28)$$

(we do not include $u^{(2k+1)}$ in our of minimal set of generators because it is not independent of the rest; indeed, $u^{(2k+1)} = \prod_{l=1}^{2k} u^{(l)}$). This shows that $G_k \cong Z_2^{\times 2k}$. It can be shown, by considering the restriction of the spinor representations of $SO(2k+1)$ to the subgroup G_k, that the cohomology class of G_k for systems in the non-trivial SPT phase with respect to $SO(2k+1)$ is that of the projective representation generated by

$$V(u^{(l)}) = \Gamma_l, \qquad (29)$$

where the $2k$ matrices Γ_l obey the anti-commuation relations $\{\Gamma_a, \Gamma_b\} = 2\delta_{a,b}$. It is straightforward to show that this cohomology class is maximally non-commutative.

Thus, we can use our general prescription [Eq. (12)] to construct a generalized KT transformation for systems in the non-trivial SPT phase with respect to $SO(2k+1)$. In analogy to the original KT transformation for the Haldane phase (which breaks the full rotation symmetry, preserving only the discrete subgroup D_2), the resulting transformed system will only have the discrete $Z_2^{\times 2k}$ symmetry instead of the full $SO(2k+1)$. Furthermore, the transformed system will be in a maximal local symmetry-breaking phase for this discrete symmetry. In this sense, the non-trivial $SO(2k+1)$ SPT phase can be understood as a result of the "hidden breaking" of the $Z_2^{\times 2k}$ symmetry. Note that, if we make a particular choice of factor system within the appropriate cohomology class, it can be shown that the generalized Kennedy-Tasaki transformation constructed according to our general prescription [Eq. (12)] coincides with the one constructed in Ref. 11.

B. SPT phases for $SU(k)$

Instead of thinking of the Haldane phase as invariant under an $SO(3)$ symmetry, we can also think of it as invariant under $PSU(2) \equiv SU(2)/\{+1, -1\}$. Of course, $PSU(2) \cong SO(3)$, but this suggests an alternative generalization of the Haldane phase: one that is invariant under an on-site representation of $PSU(k) \equiv SU(k)/C_k$, where $C_k = \{\exp(2i\pi l/k) : l = 0, \ldots, k-1\}$.

Given the definition of $PSU(k)$, we can construct a finite abelian subgroup by identifying a subgroup of $SU(k)$ which is abelian up to phase factors (i.e. up to elements of C_k). The discrete Heisenberg-Weyl group is such a subgroup; it is the group generated [in the standard representation of $SU(k)$] by the two operators

$$X = \frac{1}{W} \sum_{l=0}^{k-1} |(l+1) \bmod k\rangle\langle l|, \qquad (30)$$

$$Z = \frac{1}{W} \sum_{l=0}^{k-1} w^l |l\rangle\langle l| \qquad (31)$$

where w is a primitive k-th root of unity, and we have included the normalization factor $W = w^{(k-1)/2}$ to ensure that $\det X = \det Z = 1$. The fact that the subgroup
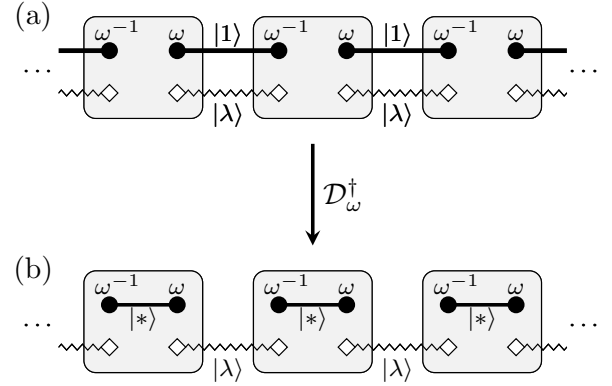

FIG. 1. (a) The "dimer state" renormalization fixed point for the SPT phase corresponding to a maximally commutative cohomology class $[\omega]$; (b) The result of applying $\mathcal{D}_\omega^\dagger$, for a particular choice of boundary conditions. Each shaded area represents one coarse-grained site. The black dots transform under irreducible projective representations with factor systems ω and ω^{-1} under the symmetry, and the diamonds do not transform at all under the symmetry. Note: Two adjacent black dots transform linearly under the symmetry; therefore, we can introduce the simultaneous eigenbasis $\{|\chi\rangle\}$ of the symmetry (with the states labelled by linear characters χ; from Schur's Lemma it follows that they must be maximally entangled). For $\chi = 1$ this gives the state $|1\rangle$ appearing in (a). In (b), we have defined $|*\rangle = \sum_\chi |\chi\rangle$. The state $|\lambda\rangle$ is not universal and depends on the specific point in the phase.

generated by X and Z is abelian up to phases follows from the relation $ZX = wXZ$. The abelian subgroup of $PSU(k)$ corresponding to the Heisenberg-Weyl group is isomorphic to $Z_k \times Z_k$. The cohomology group for $Z_k \times Z_k$ is $H^2(Z_k \times Z_k, U(1)) \cong Z_k \cong H^2(PSU(k), U(1))$, and it can be shown that the k cohomology classes of $PSU(k)$ correspond exactly to the k cohomology classes of the $Z_k \times Z_k$ subgroup. Thus, the $Z_k \times Z_k$ subgroup is sufficient to characterize all the SPT phases even in the presence of the full $PSU(k)$ symmetry. In order to apply our generalized Kennedy-Tasaki transformation, we need a maximally non-commutative cohomology class; if we let $[\omega_0]$ be a generator for the cohomology group $H^2(Z_k \times Z_k, U(1))$, then it can be shown that $[\omega_0^l]$ is maximally non-commutative if and only if l and k are coprime.

VIII. THE TOPOLOGICAL DISENTANGLER

In this section, we will briefly outline a physical interpretation of the resulting correspondence between the SPT ground states and the symmetry-breaking ground states, in terms of the entanglement structure of the ground states.

When we group sites together in blocks of size $\gg \xi$, with ξ the correlation length, any gapped ground state

starts to resemble (up to on-site unitary rotations on the blocked sites) a "dimer state" which can be viewed as a renormalization fixed point[22]. If the ground state is in an SPT phase characterized by a maximally non-commutative cohomology class, this dimer state will take the form shown in Fig. 1(a); this is a consequence of the fact that a maximally non-commutative factor system corresponds to a unique irreducible projective representation[16,23]. We see that the entanglement between two halves of the chain has two origins: the universal "topological" entanglement represented by the maximally entangled state $|1\rangle$, and the "non-topological" entanglement represented by the state $|\lambda\rangle$. (The "dot" and "diamond" particles correspond to the "protected" and "junk" subsystems discussed in Ref. 12.)

It is therefore instructive to consider what happens under $\mathcal{D}_\omega^\dagger$ to a ground state of the form shown in Fig. 1(a). The calculation required is similar to that of Sec. V. (Indeed, when written in a matrix-product state form, the state of Fig. 1(a) reduces to Eq. (15) when the "extra" particles in the $|\lambda\rangle$ state are absent.) Here we just state the result: for a suitable choice of boundary conditions, the resulting state is as depicted in Fig. 1(b); note that this resulting state is no longer invariant under the symmetry, and the orbit of this state under the symmetry is the set of symmetry-breaking ground states for the transformed system. We see that the topological component of the entanglement has been eliminated, with the non-topological part of the state remaining untouched. In this sense, we can think of the generalized KT transformation $\mathcal{D}_\omega^\dagger$ as a "topological disentangler"[24].

IX. DISCUSSION

We have presented a generalization of the Kennedy-Tasaki transformation, which maps certain one-dimensional models with SPT order to ones with traditional symmetry-breaking. This formulation further expands the characterization of SPT order as a form of hidden symmetry breaking to a broad class of models, specifically those for which the SPT order can be related to a *maximally non-commutative* factor system of a finite abelian group which acts on the system through an on-site unitary representation. Whether any analogous results hold in higher dimensions, or for other kinds of symmetries (e.g. time reversal), remains an open question. We point out, however, that in two dimensions and higher there is a different kind of duality that holds for any on-site unitary representation of a finite group G: between SPT phases and topological lattice gauge theories with gauge group G[25,26].

We have also interpreted the action of our generalized KT transformation as a *topological disentangler*[24], removing the topological component of the entanglement from the ground state. Transformations that remove entanglement from a quantum many-body system have found use in numerical methods such as the multiscale entanglement renormalisation ansatz (MERA)[27,28], and so may the generalized KT transformation presented here.

Appendix A: Choice of representative factor system

Here we will discuss the difference between $\mathcal{D}_\omega$ and $\mathcal{D}_{\omega'}$, where

$$\omega'(g,h) = \beta(g)\beta(h)\beta(gh)^{-1}\omega(g,h), \qquad \text{(A1)}$$

[for some set of phase factors $\beta(g)$] is another factor system in the same cohomology class as ω. The important thing to consider is the transformed Hamiltonians resulting from the respective transformations. Thus we will only consider the way $\mathcal{D}_\omega$ acts on symmetry-respecting observables A. First of all we observe that the isomorphism φ_ω only depends on the cohomology class, so that $\varphi_\omega = \varphi_{\omega'}$. It is then straightforward to show that

$$\mathcal{D}_{\omega'} A \mathcal{D}_{\omega'}^\dagger = b_\beta^{\otimes N} \mathcal{D}_\omega A \mathcal{D}_\omega^\dagger b_\beta^{\otimes N\dagger}, \qquad \text{(A2)}$$

where we have defined

$$b_\beta = \sum_\chi \beta\big(\varphi_\omega^{-1}(\chi)\big)|\chi\rangle\langle\chi|. \qquad \text{(A3)}$$

Therefore, the two transformed Hamiltonians differ only be a rephasing of the basis on each site. Clearly, this does not change the nature of the resulting transformed phase.

Appendix B: Proof of Eq. (23)

For simplicity of notation, we use the isomorphism φ_ω to label our site basis by group elements instead of group characters, i.e. we define

$$|g\rangle \equiv |\varphi_\omega^{-1}(g)\rangle. \qquad \text{(B1)}$$

For some choice of character χ_*, let $A, B \in \mathcal{A}_{\chi_*}$, and define $g_* = \varphi_\omega^{-1}(\chi_*)$. This implies that

$$A^\dagger = \sum_g \mu_g |g\rangle\langle gg_*| \qquad \text{(B2)}$$

$$B = \sum_g \nu_g |gg_*\rangle\langle g|. \qquad \text{(B3)}$$

for some scalars $\{\mu_g\}$ and $\{\nu_g\}$. Thus,

$$A^\dagger \otimes \mathbb{I}^{\otimes(n-2)} \otimes B = \sum_{g_1,\ldots,g_n} \mu_{g_1}\nu_{g_n}|g_1,g_2,\ldots,g_ng_*\rangle\langle g_1g_*,\ldots,g_{n-1},g_n|. \tag{B4}$$

Therefore

$$\mathcal{D}_\omega(A^\dagger \otimes \mathbb{I}^{\otimes(n-2)} \otimes B)\mathcal{D}_\omega^\dagger = \sum_{g_1,\ldots,g_n} \mu_{g_1}\nu_{g_n}\Gamma_{g_1,\ldots,g_n}|g_1,g_2,\ldots,g_ng_*\rangle\langle g_1g_*,\ldots,g_{n-1},g_n|, \tag{B5}$$

where $\Gamma_{g_1,\ldots,g_n}$ is the phase factor such that

$$V_\omega(g_ng_*)V_\omega(g_{n-1})\cdots V_\omega(g_1) = \Gamma_{g_1,\ldots,g_n}V_\omega(g_n)\cdots V_\omega(g_2)V_\omega(g_1g_*). \tag{B6}$$

This gives

$$\Gamma_{g_1,\ldots,g_n} = \omega(g_n,g_*)^{-1}\alpha_\omega(g_*,g_{n-1})\cdots\alpha_\omega(g_*,g_2)\omega(g_*,g_1), \tag{B7}$$

where $\alpha_\omega(g,h)$ is the phase factor such that

$$V_\omega(g)V_\omega(h) = \alpha_\omega(g,h)V_\omega(h)V_\omega(g). \tag{B8}$$

But now, comparing with Eq. (11), we find that

$$u(g)|h\rangle = \alpha_\omega(g,h)|h\rangle. \tag{B9}$$

Therefore, we can conclude that

$$\mathcal{D}_\omega(A^\dagger \otimes \mathbb{I}^{\otimes(n-2)} \otimes B)\mathcal{D}_\omega^\dagger = [W_\omega(g_*)A^\dagger] \otimes [u(g_*)]^{\otimes(n-2)} \otimes [BW_\omega'(g_*)^\dagger], \tag{B10}$$

which leads to Eq. (23), where we have defined

$$W_\omega(g_*) = \sum_g \omega(g_*,g)|g\rangle\langle g|, \qquad W_\omega'(g_*) = \sum_g \omega(g,g_*)|g\rangle\langle g|. \tag{B11}$$

Appendix C: Identifying phases from patterns of string order

Symmetry-breaking phases and SPT phases are two different kinds of phases that can arise in one-dimensional systems invariant under an on-site symmetry. As was shown in Ref. 8 and 20, the most general kind of phase for such systems combines both aspects. A general symmetric phase for a symmetry group G is characterized by a subgroup H (corresponding to the symmetries that are unbroken in the bulk) and a cohomology class $[\omega]$ for H, such that each of the degenerate symmetry-breaking ground states is in the SPT phase $[\omega]$ with respect to the subsymmetry. In this Appendix, we will show that, in the case of finite abelian symmetry groups G, each distinct phase gives rise to a distinct pattern of long-range string correlations, as defined through the "signature" function M of Eq. (27).

Let us first consider the case of pure SPT phases (i.e. $H = G$, and the phases are classified by cohomology classes of G). It was argued in Ref. 19 that, if the operators A and B are chosen at random, then generically one finds that $\langle C_n(A,B;g)\rangle \neq 0$ is nonzero for any ground state which is invariant under the symmetry in the bulk (which will be the case for any pure SPT phase as well

as the trivial phase). Suppose, however, that we instead choose $A, B \in \mathcal{A}_\chi$ for some character χ. Then, whenever $\chi \neq \varphi_\omega(g)$, we must be able to find some $g' \in G$ such that $\chi(g') \neq \varphi_\omega(g)$. Recalling the definition of φ_ω [Eq. (10)], and of $\mathcal{A}_\chi$ [Section VI A], this implies that $\alpha_1 \neq \alpha_2$, where α_1, α_2 are the scalars such that

$$V_\omega(g)V_\omega(g') = \alpha_1 V_\omega(g')V_\omega(g), \tag{C1}$$
$$Au(g') = \alpha_2 u(g')A \tag{C2}$$

As shown in Ref. 14, there is a selection rule which prevents the string correlation $\langle C_n(A,B;g)\rangle$ from being long-ranged when $\alpha_1 \neq \alpha_2$. On the other hand, if $\chi = \varphi_\omega(g)$ then there is no such selection rule and we expect that $\langle C_n(A,B;g)\rangle$ will generically be long-ranged even with the constraint $A, B \in \mathcal{A}_\chi$. In summary, therefore, the pure SPT phase has the signature $M(\chi,g) = 1 \Leftrightarrow \chi = \varphi_\omega(g)$. For example, if the cohomology class is maximally non-commutative, then φ_ω is invertible and thus, for each character χ, there is a unique g such that $M(\chi,g) = 1$, and vice versa. At the other extreme, if the cohomology class is trivial, then $\varphi_\omega(g) = 1$ for all g, and thus $M(\chi,g) = 1 \Leftrightarrow \chi = 1$.

The above arguments show that the possible signature functions M are in one-to-one correspondence with the homomorphisms φ_ω. In order to establish that dis-

tinct SPT phases correspond to different signature functions, it only remains to show that if ω_1 and ω_2 have different cohomology classes, then $\varphi_{\omega_1} = \varphi_{\omega_2}$. Since φ_ω is linear in ω, it suffices to prove that if $\varphi_\omega = 1$ (the trivial homomorphism), then ω has trivial cohomology class. Indeed, $\varphi_\omega = 1$ implies, by definition of φ_ω, that $V_\omega(g)V_\omega(g')V_\omega(g)^\dagger = V_\omega(g')$ for all $g, g' \in G$, which is to say all the elements $V_\omega(g)$ commute. If we choose V_ω to be irreducible, then Schur's Lemma implies that $V_\omega(g) = \beta(g)$ for some scalar phase factors $\beta(g)$. Therefore, the projective representation V_ω has trivial cohomology class.

Now let us return to the general case, where phases are classified by a subgroup $H \leq G$ and a cohomology class $[\omega]$ of H. Because all the order parameters we are considering are expectation values of symmetry-respecting operators, we just need to determine their value for a single symmetry-breaking ground state. Since each of these symmetry-breaking ground states lies in an SPT phase with respect to the subsymmetry H, we find that, for $h \in H$, $M(\chi, h) = 1$ if and only if $\chi_H = \varphi_\omega(h)$, where χ_H is the restriction χ onto the subgroup H. On the other hand, if $g \notin H$ then $M(\chi, g) = 0$ for any χ (because the symmetry operation corresponding to h is broken in the bulk[19]). In summary, the signature of a general symmetric phase is $M(\chi, g) = 1 \Leftrightarrow [g \in H \text{ and } \chi_H = \varphi_\omega(g)]$. Notice that, for $H \neq G$, there will exist non-trivial characters χ such that $\chi_H = 1$, which implies that $M(\chi, 1) = 1$. Recall that this corresponds to nonzero values of

$$\lim_{n\to\infty} \langle A^\dagger \otimes \mathbb{I}^{\otimes n} \otimes B \rangle \qquad (C3)$$

(for $A, B \in \mathcal{A}_\chi$), which is what we expect, since for $H \neq G$ there is partial symmetry-breaking, and therefore there should also be long-range order.

We will now prove that no two symmetric phases can have the same signature. Arguing as in the pure SPT case, it is easy to see that for a *fixed H* all distinct phases have different signatures. To complete the proof, we will now show that the subgroup H can be recovered from the signature, and therefore two phases with different H must have different signatures. To do this we make use of the following result:

Lemma 1. *Let G be a finite abelian group, and let H be a subgroup. Then any linear character acting on H can be extended to a linear character on G. That is, for any $\xi \in H^*$, there exists $\chi \in G^*$ such that $\chi_H = \xi$.*

Proof. Define the homomorphism $\psi : G^* \to H^*$, $\chi \mapsto \chi_H$. Observe that $\ker \psi \cong (G/H)^*$, and therefore $|\ker \psi| = |(G/H)^*| = |G/H| = |G|/|H|$. But $\psi(G^*) \cong G^*/\ker \psi$, so $|\psi(G^*)| = |G^*|/|\ker \psi| = |G|/|\ker \psi| = |H| = |H^*|$. It follows that $\psi(G^*) = H^*$, i.e. ψ is surjective. $\square$

Lemma 1 ensures that, for any $h \in H$, we can find a character $\chi \in G^*$ such that $\chi_H = \varphi_\omega(h)$, and hence $M(\chi, h) = 1$. By contrast, if $g \notin H$ then we found above that $M(\chi, g) = 0$ for all $\chi \in G^*$. Therefore, the subgroup H can be recovered from the signature according to

$$H = \{h \in G | M(\chi, h) = 1 \text{ for some } \chi \in G^*\}. \qquad (C4)$$

This completes the proof that distinct phases have distinct signatures.

[1] Xie Chen, Zheng-Cheng Gu, and Xiao-Gang Wen, "Local unitary transformation, long-range quantum entanglement, wave function renormalization, and topological order," Phys. Rev. B, **82**, 155138 (2010), arXiv:1004.3835.

[2] Marcel den Nijs and Koos Rommelse, "Preroughening transitions in crystal surfaces and valence-bond phases in quantum spin chains," Phys. Rev. B, **40**, 4709 (1989).

[3] E. Polizzi, F. Mila, and E. S. Sørensen, "$S = 1/2$ chain-boundary excitations in the Haldane phase of one-dimensional $S = 1$ systems," Phys. Rev. B, **58**, 2407 (1998).

[4] Frank Pollmann, Ari M. Turner, Erez Berg, and Masaki Oshikawa, "Entanglement spectrum of a topological phase in one dimension," Phys. Rev. B, **81**, 064439 (2010), arXiv:0910.1811.

[5] F. Verstraete, M. A. Martín-Delgado, and J. I. Cirac, "Diverging entanglement length in gapped quantum spin systems," Phys. Rev. Lett., **92**, 087201 (2004), arXiv:0311087.

[6] L. Campos Venuti and M. Roncaglia, "Analytic relations between localizable entanglement and string correlations in spin systems," Phys. Rev. Lett., **94**, 207207 (2005), arXiv:cond-mat/0503021.

[7] Xie Chen, Zheng-Cheng Gu, and Xiao-Gang Wen, "Classification of gapped symmetric phases in one-dimensional spin systems," Phys. Rev. B, **83**, 035107 (2011), arXiv:1008.3745.

[8] Norbert Schuch, David Pérez-García, and Ignacio Cirac, "Classifying quantum phases using matrix product states and projected entangled pair states," Phys. Rev. B, **84**, 165139 (2011), arXiv:1010.3732.

[9] Tom Kennedy and Hal Tasaki, "Hidden $Z_2 \times Z_2$ symmetry breaking in Haldane-gap antiferromagnets," Phys. Rev. B, **45**, 304–307 (1992); "Hidden symmetry breaking and the Haldane phase in $S = 1$ quantum spin chains," Comm. Math. Phys., **147**, 431 (1992).

[10] Frank Pollmann, Erez Berg, Ari M. Turner, and Masaki Oshikawa, "Symmetry protection of topological phases in one-dimensional quantum spin systems," Phys. Rev. B, **85**, 075125 (2012), arXiv:0909.4059.

[11] Hong-Hao Tu, Guang-Ming Zhang, and Tao Xiang, "String order and hidden topological symmetry in the $SO(2n + 1)$ symmetric matrix product states," J. Phys. A: Math. Theor., **41**, 415201 (2008), arXiv:0804.1685.

[12] Dominic V. Else, Ilai Schwarz, Stephen D. Bartlett, and Andrew C. Doherty, "Symmetry-protected phases for measurement-based quantum computation," Phys. Rev. Lett., **108**, 240505 (2012), arXiv:1201.4877.

[13] Dominic V. Else, Stephen D. Bartlett, and Andrew C. Do-

herty, "Symmetry protection of measurement-based quantum computation in ground states," New J. Phys., **14**, 113016 (2012), arXiv:1207.4805.

[14] Frank Pollmann and Ari M. Turner, "Detection of symmetry-protected topological phases in one dimension," Phys. Rev. B, **86**, 125441 (2012), arXiv:1204.0704.

[15] M Oshikawa, "Hidden $Z_2 \times Z_2$ symmetry in quantum spin chains with arbitrary integer spin," J. Phys.: Condens. Matter, **4**, 7469 (1992).

[16] Ya. G. Berkovich and E. M. Zhmud′, *Characters of Finite Groups*, Vol. 1 (American Mathematical Society, Providence, Rhode Island, 1998).

[17] I. Affleck, T. Kennedy, E.H. Lieb, and H. Tasaki, "Valence bond ground states in isotropic quantum antiferromagnets," Comm. Math. Phys., **115**, 477 (1988); Ian Affleck, Tom Kennedy, Elliott H. Lieb, and Hal Tasaki, "Rigorous results on valence-bond ground states in antiferromagnets," Phys. Rev. Lett., **59**, 799 (1987).

[18] D. Perez-Garcia, F. Verstraete, M. M. Wolf, and J. I. Cirac, "Matrix product state representations," Quant. Inf. Comput., **7**, 401 (2007), arXiv:quant-ph/0608197.

[19] D. Pérez-García, M. M. Wolf, M. Sanz, F. Verstraete, and J. I. Cirac, "String order and symmetries in quantum spin lattices," Phys. Rev. Lett., **100**, 167202 (2008), arXiv:0802.0447.

[20] Xie Chen, Zheng-Cheng Gu, and Xiao-Gang Wen, "Complete classification of one-dimensional gapped quantum phases in interacting spin systems," Phys. Rev. B, **84**, 235128 (2011), arXiv:1103.3323.

[21] Kasper Duivenvoorden and Thomas Quella, "On topological phases of spin chains," (2012), arXiv:1206.2462.

[22] F. Verstraete, J. I. Cirac, J. I. Latorre, E. Rico, and M. M. Wolf, "Renormalization-group transformations on quantum states," Phys. Rev. Lett., **94**, 140601 (2005), arXiv:quant-ph/0410227.

[23] R. Frucht, "Über die darstellung endlicher abelscher gruppen durch kollineationen," Journal für die reine und angewandte Mathematik, **1932**, 16 (1932).

[24] Kouichi Okunishi, "Topological disentangler for the valence-bond-solid chain," Phys. Rev. B, **83**, 104411 (2011), arXiv:1011.3277.

[25] Michael Levin and Zheng-Cheng Gu, "Braiding statistics approach to symmetry-protected topological phases," Phys. Rev. B, **86**, 115109 (2012), arXiv:1202.3120.

[26] Ling-Yang Hung and Xiao-Gang Wen, "Quantized topological terms in weakly coupled gauge theories and their connection to symmetry protected topological phases," (2012), arXiv:1211.2767.

[27] G. Vidal, "Entanglement renormalization," Phys. Rev. Lett., **99**, 220405 (2007), arXiv:cond-mat/0512165.

[28] G. Vidal, "Class of quantum many-body states that can be efficiently simulated," Phys. Rev. Lett., **101**, 110501 (2008), arXiv:quant-ph/0610099.